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EFFECTS OF DAMPING ON STABILITY OF ELASTIC SYSTEMS SUBJECTED TO NONCONSERVATIVE FORCES

V. V. BOLOTIN and N. I. ZHINZHER

Moscow Energetic Institute, U.S.S.R.

Abstract—A question of the correlation between stability and quasistability regions of elastic and viscoelastic systems subjected to nonconservative forces is discussed. On the base of the method of expansion in fractional powers of parameters the more rigorous considerations are presented than the considerations used in the earlier papers where the semiintuitive assumptions, the arguments by analogy and the incomplete induction methods were applied widely. In the first part of the paper a number of general statements concerning both continuous and discrete systems are proved. It is shown that for real laws of damping a considerable part of the quasistability region belongs in fact to instability region. From this point of view a number of papers dealing with nonconservative stability problems (including panel flutter problems) must be reconsidered. To illustrate the general statements, in the second part of the paper a numerical examination of stability of cantilever bar made of the standard viscoelastic solid and subjected to follower and dead forces is presented. Some phenomena inherent to the nonconservative viscoelastic systems are discussed.

1. INTRODUCTION

STABILITY of elastic systems subjected to nonconservative forces has been studied by many authors beginning with Nikolai [1, 2]. A review of results in this area is given in a book by Bolotin [3] and in papers by Herrmann [4] and Dzhanelidze [5].

One of the most important and interesting aspects of the theory is connected with damping effects in stability problems. Ziegler [6], dealing with a double mathematical pendulum subjected on a free end to a follower (tangential) force, has found that addition of a small damping can reduce the value of the critical force in comparison with one found not taking into account the damping. Bolotin [7] has investigated a dependence of critical parameters on the ratio of partial damping coefficients. He has proved that a reduction of critical parameters caused by the addition of vanishing damping is absent only in a case of equal partial coefficients. Later, effects of damping on stability of elastic systems subjected to nonconservative forces have been studied by many authors. Some of the problems have been discussed by Leonov and Zorii [8], Herrmann and Jong [9, 22] and Leipholz [10]. Leonov and Zorii have studied effects of friction on the stability of a cantilever bar subjected to follower and dead forces and having at its end two equal masses located at some distance from each other. Herrmann and Jong [9] studied the influence of damping in Ziegler's model [6]. Leipholz [10] has considered a discrete system with a diagonal damping matrix.

Some general results related to the stability of nonconservative systems have been obtained by Nemat-Nasser, Prasad and Herrmann [11], Nemat-Nasser and Herrmann [12]. Expanding the characteristic determinant in integer powers of damping parameters, authors have made some conclusions about the behaviour of characteristic exponents of discrete systems with small damping. Their results are presented in form of theorems on the destabilization condition due to damping. An attempt was made to extend these results to continuous systems. This question was also discussed by Nemat-Nasser [24].

The expansions used in papers [11, 12] become improper near the critical values of parameters of external forces. Zhinzher [13] has applied in this case an expansion in fractional powers of the damping parameters. Thus some general statements on the behaviour of critical parameters of external forces when damping vanishes are obtained.

This paper is dedicated to a systematical study of damping effects on the stability of finite-degree-of-freedom and continuous systems subjected to nonconservative forces. A method of expansion in fractional powers of parameters is applied. This method permits one to establish in a general form a number of results which have been discussed previously only on the base of simple particular examples. Further by means of this method a set of general statements is proved rigorously.

The paper consists of two parts. The first part deals with general problems. Generalization and development of some results [3, 6–13] concerning to the paradoxical behaviour of elastic systems in the presence of small damping is presented. This question is discussed from the point of view of the Liapunov's theory of stability. A concept of quasistability corresponding to the doubtful case in the Liapunov's theory and a concept of quasicritical load's parameters are introduced. The not very appropriate concept of the destabilizing effect of damping is replaced by more rigorous and exact terms. In a general form an investigation is undertaken of an equation connecting characteristic exponents, damping- and load's parameters. A question of the structure of an expansion of the roots of the characteristic equation in fractional powers of the damping parameters is investigated. General properties of critical parameters in the presence of small damping and their relations with quasicritical parameters are found. The cases of divergence instability (zero characteristic exponents) and cases of flutter instability are considered. An example of elastic system with multiple natural frequencies first considered by Nikolai [1, 2] is discussed separately. General methods are illustrated with examples of discrete and continuous elastic systems.

The second part of the paper is dedicated to a comprehensive numerical analysis of one special problem. As such one the stability problem of a cantilever bar subjected to tangential and dead forces is considered. The material of the bar is supposed to be a standard viscoelastic solid. So, damping forces are characterized by means of two constants; it allows to include a wide diapason of variation of partial damping coefficients. The two damping constants and two parameters of external forces form a four-dimensional space. The stability region in this space (more exactly in correspondingly selected subspace) is determined by the mapping of imaginary axes on the characteristic exponents' plane. Calculations were made by digital computer. The dependence of critical parameters on damping parameters and on the ratio between parameters of the tangential and the dead forces is discussed. The jump paradox of the resultant critical force discovered firstly by Dzhanelidze [17] is also discussed. The behaviour of characteristic exponents due to variation of external forces parameter and damping parameter is investigated.

2. GENERAL CONCEPTS

The fundamental problem of the stability theory of deformable systems is the calculation of values of system parameters and (or) external condition parameters corresponding to transition from stability to instability. These values are called *critical*. Most often the values of external forces are considered as such parameters; in this case the critical forces are discussed.

Let us consider for example a problem characterized by a single parameter β. Without any limitation of generality, we can assume that β is varying within the limits $0 \leq \beta < \infty$ and that the undisturbed motion is stable at $\beta = 0$. The upper limit of the values $\beta = \beta_*$ when the undisturbed motion is stable is called critical. In the more general case of finite number of parameters it is expedient to introduce the n-dimensional space of parameters $\beta_1, \beta_2, \ldots, \beta_n$ and distinguish in this space stability and instability regions. The surfaces $\Phi(\beta_1, \beta_2, \ldots, \beta_n) = 0$ dividing the stability and instability regions are called critical.

A general investigation method of stability of elastic systems consists of the analysis of the set of motions neighbouring to the undisturbed one. This method connected with the general theory of stability is called the dynamic one [3]. The linearized equations of disturbed motion are usually applied by investigation of stability of equilibriums' forms. These equations describe small vibration of system near the undisturbed equilibrium. So this method is called also small vibrations' method [1, 2].

The small vibration method is analogous to the well known method of "equations aux variations" in classical stability theory of Liapunov–Poincaré. In this theory theorems on the conditions under which the linearized equations give the comprehensive solution of the Liapunov's stability problem are proved rigorously. The generalization of the Liapunov's theory to continuous systems is based on the consideration of solutions' closeness in some metric functional spaces. Up to now all rigorously proved results in this field concerned potential systems only [15, 16]. Nevertheless with some restrictions we can expect that a decaying character of all possible motions of the linearized system would provide asymptotic stability of nonlinear system in some properly chosen functional space. This statement turns out to be true in particular for elastic and viscoelastic systems of finite dimension. In this paper we shall assume this assertion as a postulate.

Let us consider the problem of the stability of equilibrium when the elastic system is subjected to nonconservative forces depending on position and to dissipative forces. The linearized equations of motion of the elastic system are satisfied, if the displacement vector u is chosen in the form

$$u = \varphi\, e^{st}. \tag{2.1}$$

Here φ is the vector defining the vibration mode, t — time, s — characteristic exponent. The vector φ is to be determined from the following nonselfadjoint boundary value problem [3]:

$$[As^2 + C + \varepsilon D(s) + \beta B]\varphi = 0. \tag{2.2}$$

In the operator equation (2.2) A and C are positive selfadjoint linear operators in some Hilbert's space (gradients of kinetic and potential energy of elastic system respectively), $D(s)$ is a positive operator characterizing damping forces, B is a nonselfadjoint linear operator in the same space, characterizing nonconservative forces. In the general case the operator of nonconservative forces depends on the characteristic exponent s. When the material is viscoelastic, the operator C depends on s too. The dependence of the operators D and B on the parameters ε and β can be more complicated, but the operators will be zero operators when ε and β vanish. The specific form of the operators is determined by the linearized vibration's equations and by the boundary conditions.

The operator equation (2.2) corresponds to an eigenvalue problem with three parameters: the characteristic exponent s, the load parameter β and the damping parameter ε. When $\beta = 0$ all the characteristic exponents are in the left half-plane of the complex variable.

It is because the operators A, C and D are positive. All the motions (2.1) of the linearized system are decaying in time. Analogous to the classical stability theory this will be qualified as asymptotic stability. When the parameter β varies continuously, the characteristic exponents vary continuously too. At some value β at least one of the exponents will be imaginary. When parameter β increases, the characteristic exponents with positive real part appear and consequently the instability occurs (Fig. 1a). The critical value β_* is determined from condition that at $\beta > \beta_*$ among the characteristic exponents s at least one exponent with a positive real part will appear. When the transition to the right half-plane through the value $s = 0$ takes place, so instability is unvibrational. In other cases the vibrational instability takes place. In aeroelasticity problems one speaks of divergence and flutter, respectively.

When the dissipative forces are not considered we have instead of equation (2.2) an operator equation

$$[As^2 + C + \beta B]\varphi = 0. \tag{2.3}$$

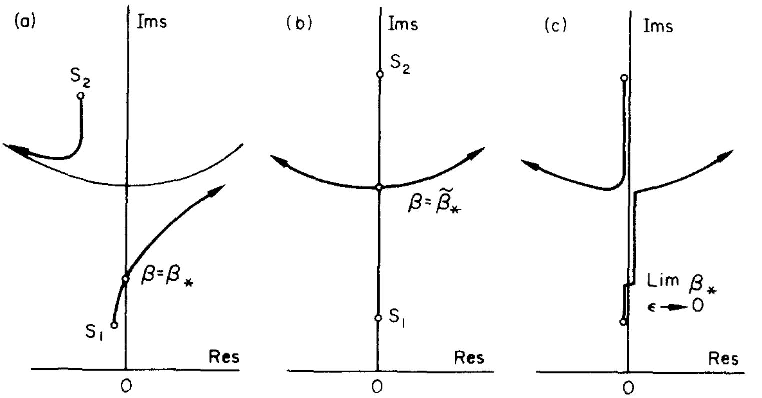

FIG. 1. Behaviour of the characteristic exponents on the complex plane.

If $\beta = 0$ the boundary value problem (2.3) is selfadjoint. All the eigenvalues s^2 are negative. Hence all the characteristic exponents are purely imaginary. Let the parameter β increase monotonously. At some value $\beta = \tilde{\beta}_*$ one or several couples of exponents become multiple. When the parameter β increases further the exponents can become complex. Thus the characteristic exponents with positive real parts appear (Fig. 1,b).

In papers dealing with stability of elastic systems the presence of characteristic exponents on the imaginary axis is qualified usually as stability, and the value $\tilde{\beta}_*$ as a critical one. In other words, stability is interpreted as a bounded character of motions of the linearized system, near equilibrium. This is analogous to the doubtful (critical) case in Liapunov's theory, when "equations of variations" do not answer the question about stability. Thus there is no sufficient reason for application of the small vibration method in this version. Even an analogy with the stability theory of discrete systems does not hold.

Analogous to the classical stability theory the case of purely imaginary characteristic exponents is to be qualified as a doubtful one. When the external forces are potential, an application of direct Liapunov's method in some metric spaces yields a rigorous proof of stability at $\beta < \tilde{\beta}_*$ [15, 16]. In this case an introduction of arbitrary small (but complete) dissipation displaces all the characteristic exponents from the imaginary axis to the left halfplane. If the external forces are nonconservative, the addition of the dissipative forces with complete dissipation eliminates the critical case too. But it appears [6, 7, 9, 22] that, generally, only a part of the segment $0, \tilde{\beta}_*$ belongs to the stability region and that the vanishing of the dissipation does not imply $\beta_* \to \tilde{\beta}_*$ (Fig. 1c).

In previous papers dealing with the damping effects on stability of nonconservative systems the inequality $\beta_* < \tilde{\beta}_*$ has been interpreted as a destabilization phenomenon caused by damping. But it seems, that it would be more correct to speak not of destabilization but of false conclusions based on the interpretation of the critical case as a stable one. The "destabilization paradoxes" are consequences of noncritical application of the small vibrations' method. We shall show further below that solutions obtained without a consideration of damping, retain some sense. It is expedient to introduce a terminology, providing an appropriate place for these solutions. Let us call quasistability a case, when all the characteristic exponents are on the imaginary axis. Let us call quasicritical such a value of $\tilde{\beta}_*$, that as $\beta > \tilde{\beta}_*$ at least one exponent comes to the right halfplane. Using this terminology, we can say that in papers discussing stability without considerations of damping, in fact only the quasicritical parameters have been determined. A question appears: what is the correlation between the critical and quasicritical values? In the real structures the damping is sufficiently small usually. Hence another question appears: how the critical values behave, when parameters of dissipation approach to zero? These questions will be discussed in the general form applicable both for discrete and continuous systems.

3. EXPANSION OF CHARACTERISTIC EXPONENTS IN FRACTIONAL POWERS OF PARAMETERS

Let the characteristic equation of the eigenvalue problem (2.2) be

$$F(s^2, \varepsilon s, \beta) = 0. \tag{3.1}$$

Setting $\varepsilon = 0$ we obtain a characteristic equation of the eigenvalue problem (2.3). At $\varepsilon = 0$ and $\beta = \tilde{\beta}_*$ the equation (3.1) has n-multiple root $s = s_*$. Hence

$$F(s_*^2, 0, \tilde{\beta}_*) = 0, \qquad \partial^k F(s_*^2, 0, \tilde{\beta}_*)/\partial s^k = 0 \quad (k < n), \qquad \partial^n F(s_*^2, 0, \tilde{\beta}_*)/\partial s^n \neq 0.$$

The quasicritical value $\tilde{\beta}_*$ for the undamped elastic system corresponds to the junction of n couples of the characteristic exponents. The left part of the equation (3.1) is an analytical function. At arbitrary fixed β this equation determines inexplicit function $s(\varepsilon)$. According to the preparative theorem of Weierstrass [18], this equation at $\beta = \tilde{\beta}_*$ and in the neighbourhood of the point $s = s_*$, $\varepsilon = 0$ is equivalent to an equation of power n for $s - s_*$. Coefficients of the last equation are analytic functions of ε. In the neighbourhood of the point $\varepsilon = 0$ not more than n branches exist. These branches are expanded in fractional powers of ε

$$s_j(\varepsilon) = s_* + \sum_{l=1}^{\infty} c_l \varepsilon^{\alpha_l} \qquad (j \leq n, \alpha_{l_1} > \alpha_{l_2} \quad \text{at} \quad l_1 > l_2) \tag{3.2}$$

and this expansion converges near $\varepsilon = 0$. The power of the first term of expansion is larger or equal to $1/n$.

For the calculation of these expansions let us present the analytical function at $\beta = \tilde{\beta}_*$ in the neighbourhood $s = s_*$, $\varepsilon = 0$ by the series

$$F(s^2, \varepsilon s, \tilde{\beta}_*) = \sum_{j=0}^{n} \sum_{k=0}^{m} a_{jk}(s-s_*)^j \varepsilon^k + \dots, a_{jk} = \frac{1}{j!k!} \frac{\partial^{j+k} F(s_*^2, 0, \tilde{\beta}_*)}{\partial s^j \partial \varepsilon^k}. \tag{3.3}$$

The dots denote terms of the order of magnitude $o(|r|^{m+n})$, $r^2 = |s-s_*|^2 + \varepsilon^2$, $a_{00} = 0$. Substituting the series $s = s_* + c_1 \varepsilon^{\alpha_1} + \dots$ into the equation (3.3), we get an identity. For determination of an unknown exponent α_1 let us construct the Newton's polygon [19]. On the x, y plane we plot the points with coordinates (j, k) and construct a convex broken line through the extreme left points. Unknown exponents α_1 are equal to tangents of the angles between the segments of broken line nonparallel to the coordinate axes, and the x-axis. Let the extreme points of the segment be (j, k) and (j_1, k_1), then c_1 is to be determined by the equation $a_{j_1 k_1} c_1^{j_1 - j} + \dots + a_{jk} = 0$. The power of the equation is equal to the quantity of units contained in the projection of this segment on the x-axis. It is not more than n. For the calculation of the following term in the expansion (3.2) we must put in equation (3.3) $s = s_* + c_1 \varepsilon^{\alpha_1} + w$ and repeat all the procedure for new series, etc.

Above the equation (3.1) has been considered, when β is fixed. It is easy to reproduce corresponding considerations for a case when ε is fixed. Taking an implicit function $s(\beta)$, we obtain instead of the expansions (3.2) the following expansion suitable for the neighbourhood of the point $\beta = \tilde{\beta}_*$:

$$s_j(\beta - \tilde{\beta}_*) = s_* + \sum_{l=1}^{\infty} c_l'(\beta - \tilde{\beta}_*)^{\alpha_l} \qquad (j \leq n, \alpha_{l_1} > \alpha_{l_2} \quad \text{at} \quad l_1 > l_2). \tag{3.4}$$

Instead of the expression (3.3) we get

$$F(s^2, 0, \beta) = \sum_{j=0}^{n} \sum_{k=0}^{m} a_{jk}'(s-s_*)^j (\beta - \tilde{\beta}_*)^k + \dots, \tag{3.5}$$

where dots denote terms of order of magnitude $o(|r|^{m+n})$, $r^2 = |s-s_*|^2 + (\beta - \tilde{\beta}_*)^2$, $a_{00}' = 0$.

Using expansions (3.2) and (3.3), it is possible to prove a number of statements about the relation between the parameters β_* and $\tilde{\beta}_*$, when damping approaches zero. If at least one of the branches (3.2) at arbitrarily small $\varepsilon > 0$ has a positive real part, so the statement that $\beta_* < \tilde{\beta}_*$ as $\varepsilon \to 0$ is true. If all the branches have negative real parts, then $\beta_* = \tilde{\beta}_*$ as $\varepsilon \to 0$. In the case of vanishing damping it is sufficient to take into consideration only first terms of expansions with real parts not equal to zero.

Let us prove firstly that at

$$\partial F(s_*^2, 0, \tilde{\beta}_*)/\partial \beta \neq 0,$$

the multiplicity of the root s_* may be not more than two. In fact, in this case

$$a_{01}' = \partial F(s_*^2, 0, \tilde{\beta}_*)/\partial \beta \neq 0, \qquad a_{n0}' = \partial^n F(s_*^2, 0, \tilde{\beta}_*)/\partial s^n \neq 0.$$

Hence

$$\alpha_1 = \frac{1}{n}, \qquad c_1 = \left(-\frac{a_{01}'}{a_{n0}'}\right)^{1/n}, \qquad s(\beta - \tilde{\beta}_*) = s_* + \left[-\frac{a_{01}'(\beta - \tilde{\beta}_*)}{a_{n0}'}\right]^{1/n} + \dots. \tag{3.6}$$

As the boundary value problem (2.3) for $\varepsilon = 0$ and $\beta < \tilde{\beta}_*$ has negative eigenvalues, so the second term of the series must be purely imaginary. The last fact is possible only if $n = 2$. Further we shall be limited by the case $n = 2$.

4. RELATIONS BETWEEN CRITICAL AND QUASICRITICAL PARAMETERS

We consider at first a case of flutter instability. In this case $s_* = i\omega_* \neq 0$. Let us show, that in the presence of external damping

$$\beta_* = \tilde{\beta}_* \quad \text{at} \quad \varepsilon \to 0. \tag{4.1}$$

In fact, in the case of external damping the damping operator in equation (2.2) is $D(s) = As$. The characteristic equation (3.1) at $\beta = \tilde{\beta}_*$ has a form

$$F(s^2+\varepsilon s, \tilde{\beta}_*) = 0. \tag{4.2}$$

We get from equation (4.2)

$$a_{01} = a_{10} = 0, \qquad a_{02} = \frac{s_*^2}{2}\frac{\partial^2 F(s_*^2, \tilde{\beta}_*)}{\partial x^2}, \qquad a_{11} = 2s_*^2\frac{\partial^2 F(s_*^2, \tilde{\beta}_*)}{\partial x^2},$$

$$a_{20} = 2s_*^2\frac{\partial^2 F(s_*^2, \tilde{\beta}_*)}{\partial x^2},$$

where $x = s^2+\varepsilon s$. The power of the first term of the expansion $\alpha_1 = 1$ and coefficients of the expansion are to be determined from the equation $a_{20}c_1^2+a_{11}c_1+a_{02} = 0$. Taking into account the explicit expressions of these coefficients, we obtain $4c_1^2+4c_1+1 = 0$. Roots of this equation are $c_{1(1,2)} = -1/2$. Hence

$$s_{1,2}(\varepsilon) = s_*-\tfrac{1}{2}\varepsilon+\ldots$$

and the relation (4.1) is valid.

If the coefficient

$$a_{01} = \partial F(s_*^2, 0, \tilde{\beta}_*)/\partial\varepsilon \neq 0,$$

so the relation

$$\beta_* < \tilde{\beta}_* \quad \text{for} \quad \varepsilon \to 0, \tag{4.3}$$

holds. In fact, in this case in the neighbourhood of the point $\varepsilon = 0$ we have two simple branches

$$s_{1,2}(\varepsilon) = s_*+\left(-\frac{a_{01}}{a_{20}}\right)^{1/2}\varepsilon^{1/2}+\ldots. \tag{4.4}$$

From equation (3.1) we get

$$a_{01} = s_*\frac{\partial F(s_*^2, 0, \tilde{\beta}_*)}{\partial y}, \qquad a_{20} = 2s_*^2\frac{\partial^2 E(s_*^2, 0, \tilde{\beta}_*)}{\partial x^2}, \tag{4.5}$$

where $x = s^2$, $y = \varepsilon s$. As $s_* = i\omega$ and ω_* and all the derivatives in the expressions (4.5) are real quantities, so one of the branches (4.4) has a positive real part. Therefore the relation (4.3) is proved.

Let us come to a case of divergence instability ($s_* = 0$). We suppose that the elastic system is subjected to two sets of forces given with parameters β_1 and β_2. At some combination of these parameters the divergence instability may occur. The characteristic equation has a form

$$F(s^2, \varepsilon s, \beta_1, \beta_2) = 0. \tag{4.6}$$

We show that in the presence of the external damping only, the relation similar to (4.1) takes place too. From an equation analogous to the equation (4.2) we get

$$\partial^k F/\partial \varepsilon^k = s^k \partial^k E/\partial x^k, \qquad k = 1, 2, \ldots.$$

Therefore

$$F(s^2 + \varepsilon s, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}) = s \sum_{j=0}^{n} \sum_{k=0}^{m} a_{jk} s^{j-1} \varepsilon^k + \ldots.$$

One of the characteristic exponents does not depend on ε, i.e. $s_1 \equiv 0$. The calculations give

$$a_{20} = \partial F(0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*})/\partial x, \qquad a_{11} = \partial F(0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*})/\partial x.$$

Hence the power of the first term in the expansion $\alpha_1 = 1$, and the coefficients are to be determined from the equation $a_{20}c_1 + a_{11} = 0$. So $c_1 = 1$ and $s_2(\varepsilon) = -\varepsilon + \ldots$, which proves the relation (4.1).

In the general case of damping forces a following statement is true: the relation (4.1) takes place when and only when the coefficient $a_{11} \neq 0$ and $a_{11}a_{20} > 0$. The condition $a_{11} = 0$ yields an equation of curves where the divergence and flutter instability's critical surfaces intersect. These curves on the divergence instability's surfaces we call singular. In the case, when the parameter's space is two-dimensional, we speak of singular points.

Differentiation of the equation (4.6) gives

$$\partial^k F/\partial \varepsilon^k = s^k \partial^k F/\partial y^k, \qquad k = 1, 2, \ldots.$$

Hence

$$F(s^2, \varepsilon s, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}) = s \sum_{j=0}^{n} \sum_{k=0}^{m} a_{jk} s^{j-1} \varepsilon^k + \ldots,$$

and $s_1 \equiv 0$. The calculations give

$$a_{11} = \frac{\partial F(0, 0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*})}{\partial y}, \qquad a_{20} = \frac{\partial F(0, 0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*})}{\partial x}.$$

Then

$$\alpha_1 = 1, \qquad c_1 = -\frac{a_{11}}{a_{20}}, \qquad s_2(\varepsilon) = -\frac{a_{11}}{a_{20}}\varepsilon + \ldots.$$

If $a_{11}a_{20} > 0$ (the coefficients are real) the relation (4.1) holds.

Now we consider a new function

$$F_1(s^2, \varepsilon s, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}) = \frac{1}{s} F(s^2, \varepsilon s, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}).$$

For this function at $\varepsilon = 0$ the root $s = 0$ is simple. According to the theorem about implicit functions, this root may be expanded in power series convergent in some neighbourhood of the point $\varepsilon = 0$:

$$s_2(\varepsilon) = s_0 + \sum_{k=1}^{\infty} a_k \varepsilon^k, \qquad s_0 = s(0), \qquad a_k = \frac{1}{k!}\frac{d^k s(0)}{d\varepsilon^k}. \tag{4.7}$$

It is easy to show that

$$a_2 = a_3 = 0, \qquad s_2(\varepsilon) = a_1 \varepsilon + 0(\varepsilon^4).$$

When damping is sufficiently small, we may take into consideration the first term only. At $a_1 > 0$ this branch always has a positive real part, and at $a_1 < 0$ has a negative one. Equation $a_1 = 0$ gives the singular (points) curves.

5. EQUATION OF THE CRITICAL SURFACE IN A CASE OF INFINITESIMAL DAMPING

At $\beta < \tilde{\beta}_*$ the equation (3.1) has no multiple roots. An arbitrary root of this equation may be expanded in power series of the type (4.7). Coefficients of this series obviously may be expressed by means of partial derivatives of the left part of the equation (3.1). These derivatives must be taken at $\varepsilon = 0$, and they depend on s_0^2 and on parameters of the external forces. Only the case $s_0^2 \neq 0$ is interesting. From the properties of eigenvalues of boundary value problem (2.2) it follows, that for a wide class of the damping forces' operators, the coefficient a_1 is rigorously negative at $\beta = 0$. Using continuity considerations, the critical values of parameters at $\varepsilon \to 0$ are to be determined from equation $a_1(s_0^2, \beta_1, \beta_2 \ldots, \beta_n) = 0$. The equation (3.1) yields

$$a_1(s_0^2, \beta_1, \beta_2, \ldots, \beta_n) = -\frac{\partial F(s_0^2, 0, \beta_1, \beta_2, \ldots, \beta_n)/\partial\varepsilon}{\partial F(s_0^2, 0, \beta_1, \beta_2, \ldots, \beta_n)/\partial s}.$$

Putting this coefficient to be equal to zero we obtain

$$\frac{\partial F(s_0^2, 0, \beta_1, \beta_2, \ldots, \beta^n)}{\partial\varepsilon} = 0. \tag{5.1}$$

As s_0^2 are eigenvalues of undamped system, so the equation (5.1) must be supplemented by corresponding equation. As a result we have

$$\frac{\partial F(s_0^2, 0, \beta_1, \beta_2, \ldots, \beta_n)}{\partial\varepsilon} = 0, \qquad F(s_0^2, 0, \beta_1, \beta_2, \ldots, \beta_n) = 0. \tag{5.2}$$

Thus the equation of critical flutter surfaces at infinitesimal damping is

$$R\left[\frac{\partial F(s^2, 0, \beta_1, \beta_2, \ldots, \beta_n)}{\partial\varepsilon}, \qquad F(s^2, 0, \beta_1, \beta_2, \ldots, \beta_n)\right] = 0, \tag{5.3}$$

where R is a resultant. In general case construction of the resultant of two integer transcendental functions is impossible. Numerical solution of this problem does not meet any complications. In presence of the external damping only we have

$$\frac{\partial F(s_0^2, \beta_1, \beta_2, \ldots, \beta_n)}{\partial\varepsilon} = \frac{1}{2}\frac{\partial F(s_0^2, \beta_1, \beta_2, \ldots, \beta_n)}{\partial(s^2)},$$

and the formula (5.3) becomes

$$R\left[\frac{\partial F(s^2,\beta_1,\beta_2,\ldots,\beta_n)}{\partial(s^2)}, F(s^2,\beta_1,\beta_2,\ldots,\beta_n)\right] = D[F(s^2,\beta_1,\beta_2,\ldots,\beta_n)] = 0.$$

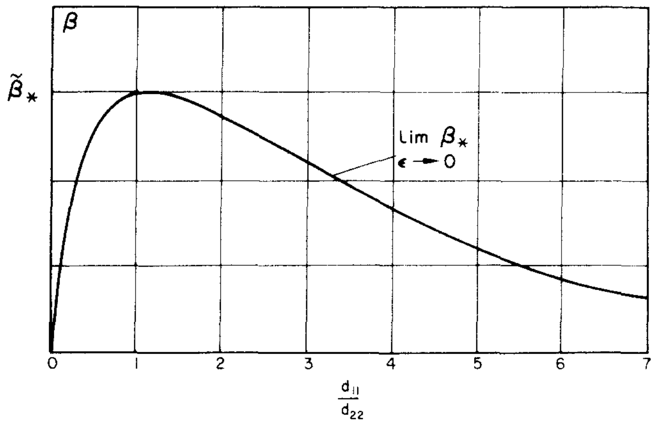

FIG. 2. Critical value as a function of the ratio of partial damping coefficients.

The equation of critical surfaces is obtained by putting to zero a discriminant of the characteristic equation

$$D(\beta_1,\beta_2,\ldots,\beta_n) = 0. \tag{5.4}$$

This result coincides with results about the correlation between the critical and quasicritical parameters in the presence of external damping only obtained earlier.

6. APPLICATION OF THE THEORY TO FINITE-DEGREE-OF-FREEDOM SYSTEMS

The characteristic equation of an n-degree-of-freedom system in the presence of dissipative forces is [3]

$$\Delta(s^2,\varepsilon s,\beta) = |(s^2+\Omega_j^2)\delta_{jk}+\varepsilon s d_{jk}+\beta\Omega_j^2 b_{jk}| = 0. \tag{6.1}$$

Here Ω_j are the partial natural frequencies, $[d_{jk}]$ and $[b_{jk}]$ are the matrices—finite-dimensional analogs of operators $C^{-1}D$ and $C^{-1}B$ respectively. The matrix analog of the finite-dimensional operator $C^{-1}A$ is reduced to the diagonal form.

The statements about the correlation between the critical and quasicritical values, which has been obtained earlier, may be paraphrased in terms of the expression (6.1). The simplest form of these statements will be obtained if the dissipation matrix is diagonal. We discuss only this case.

Let us consider the flutter instability ($s_* \neq 0$, $\tilde{\beta}_* \neq 0$). From the expression (6.1) we obtain

$$\begin{aligned} a_{01} &= s_* d_{jj} c^{jj}(s_*^2, 0, \tilde{\beta}_*), \qquad a_{10} = 2 s_* \delta_{jj} c^{jj}(s_*^2, 0, \tilde{\beta}_*) \\ a_{02} &= \frac{1}{2} s_* d_{jj} \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial \varepsilon}, \\ a_{11} &= d_{jj} c^{jj}(s_*^2, 0, \tilde{\beta}_*) + s_* d_{jj} \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial s}, \\ a_{20} &= \delta_{jj} c^{jj}(s_*^2, 0, \tilde{\beta}_*) + s_* \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial s}, \end{aligned} \tag{6.2}$$

where c^{jk} are algebraic supplements of the corresponding elements in the determinant (6.1).

When the diagonal elements of the dissipation matrix are equal, so $\beta_* = \tilde{\beta}_*$ at $\varepsilon \to 0$. This phenomenon was discovered by Bolotin [7]. Let us prove this result using general considerations. Let $d_{jj} = d_{kk} = d$. Without limiting of generality we can assume $d = 1$. As by the condition $a_{10} = 0$ so $a_{01} = s_* \delta_{jj} c^{jj} = 0$. Other coefficients become

$$\begin{aligned} a_{02} &= \frac{1}{2} s_* \delta_{jj} \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial \varepsilon}, \\ a_{11} &= s_* \delta_{jj} \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial s}, \\ a_{20} &= s_* \delta_{jj} \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial s}. \end{aligned} \tag{6.3}$$

It is easy to show that in this case

$$\frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial s} = 2 \frac{\partial c^{jj}(s_*^2, 0, \tilde{\beta}_*)}{\partial \varepsilon}.$$

Hence $\alpha_1 = 1$, the roots of the equation $a_{20} c_1^2 + a_{11} c_1 + a_{02} = 0$ have negative real parts, and the statement is proved.

If the sum $d_{jj} c^{jj}$ is not equal to zero, then the coefficient a_{01} is purely imaginary. The coefficient a_{20} is always real, and one of the branches (4.4) has a positive real part. Therefore the relation $\beta_* < \tilde{\beta}_*$ for $\varepsilon \to 0$ holds.

We consider separately the case of double partial frequencies. Without limiting the generality let us assume that $\Omega_1 = \Omega_2 = \Omega_0 = i s_*$. At beginning we examine the behaviour of the characteristic exponents of the undamped systems when $\beta > 0$. From the equation (6.1) for $\varepsilon = 0$ we get

$$\Delta(s^2, \beta) = |(s^2 + \Omega_j^2)\delta_{jk} + \beta \Omega_j^2 b_{jk}| = 0. \tag{6.4}$$

Let us suppose for example that $b_{jk} b_{kj} < 0$, $b_{kk} = 0$. Calculations yield

$$\begin{aligned} a'_{01} &= 0, \qquad a'_{11} = 0, \\ a'_{20} &= -4\Omega_0^2 \prod_{\substack{j \neq 1 \\ j \neq 2}}^{n} (\Omega_j^2 - \Omega_0^2), \\ a'_{02} &= |b_{12} b_{21}| \prod_{\substack{j \neq 1 \\ j \neq 2}}^{n} (\Omega_j^2 - \Omega_0^2). \end{aligned}$$

Hence for arbitrary $\beta > 0$ one of the characteristic exponents is on the right halfplane. Thus $\tilde{\beta}_* = 0$.

Now let us examine the dependence of the characteristic exponents on the parameter ε for the case $\tilde{\beta}_* = 0$. From equation (6.1) we get

$$a_{01} = 0, \qquad a_{11} = -2\Omega_0^2(d_{11}+d_{22}) \prod_{\substack{j\neq 1\\ j\neq 2}}^{n} (\Omega_j^2-\Omega_0^2),$$

$$a_{02} = -\Omega_0^2 \begin{vmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{vmatrix} \prod_{\substack{j\neq 1\\ j\neq 2}}^{n} (\Omega_j^2-\Omega_0^2), \qquad a_{20} = -4\Omega_0^2 \prod_{\substack{j\neq 1\\ j\neq 2}}^{n} (\Omega_j^2-\Omega_0^2).$$

Here $\alpha_1 = 1$, and the equation $a_{20}c_1^2+a_{11}c_1+a_{02} = 0$ has roots with negative real parts. Therefore $\beta_* = \tilde{\beta}_*$ at $\varepsilon \to 0$.

Now we come to the case of divergence instability ($s_* = 0$). We present the characteristic equation for a system with n degrees of freedom in a form

$$\Delta(s^2, \varepsilon s, \beta_1, \beta_2) = |(s^2+\Omega_1^2)\delta_{jk}+\varepsilon s d_{jk}+\Omega_j^2(\beta_1 b_{jk}^{(1)}+\beta_2 b_{jk}^{(2)})| = 0, \tag{6.5}$$

where β_1 and β_2 are parameters of the external forces. Let us suppose that a divergence instability of the system is possible, i.e. there exist such quantities $\tilde{\beta}_{1*}$ and $\tilde{\beta}_{2*}$ that

$$\Delta(0, 0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}) = 0. \tag{6.6}$$

Otherwise we can write

$$|\delta_{jk}+\tilde{\beta}_{1*}b_{jk}^{(1)}+\tilde{\beta}_{2*}b_{jk}^{(2)}| \prod_{j=1}^{n} \Omega_j^2 = 0. \tag{6.7}$$

Now we examine the dependence of the roots of the equation (6.5) at $\beta_1 = \tilde{\beta}_{1*}$ and $\beta_2 = \tilde{\beta}_{2*}$ on damping parameter ε. From the equation (6.5) we obtain

$$\begin{aligned} &a_{0k} = 0 \quad (k = 0, 1, 2, \ldots), && a_{10} = 0, \\ &a_{20} = \delta_{jj}c^{jj}(0, 0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}), && a_{11} = d_{jk}c^{jk}(0, 0, \tilde{\beta}_{1*}, \tilde{\beta}_{2*}). \end{aligned} \tag{6.8}$$

All the coefficients are real. From the first condition (6.8) it follows that the polynomial (6.5) has a factor s. Thus $s_1 \equiv 0$. The remaining conditions yield $\alpha_1 = 1$, $c_1 = a_{11}/a_{20}$ and

$$s_2(\varepsilon) = -\frac{a_{11}}{a_{20}}\varepsilon + \ldots. \tag{6.9}$$

Consequently, if $a_{11}a_{20} > 0$, so $\beta_{1*} = \tilde{\beta}_{1*}, \beta_{2*} = \tilde{\beta}_{2*}$ at $\varepsilon \to 0$; if $a_{11}a_{20} < 0$, so $\beta_{1*} < \tilde{\beta}_{1*}$ or $\beta_{2*} < \tilde{\beta}_{2*}$, at $\varepsilon \to 0$. Singular points are to be determined from the condition $a_{11} = 0$.

Let us construct an equation of critical flutter surface for a case of infinitesimal damping. Developing the determinant, we write the equation (6.1) in a form

$$P_{2n} = p_0 s^{2n}+\varepsilon p_1 s^{2n-1}+p_2 s^{2n-2}+\varepsilon p_3 s^{2n-3}+\ldots+\varepsilon p_{2n-1}s+p_{2n}+\ldots. \tag{6.10}$$

Unwritten terms have an order of magnitude $o(\varepsilon)$; the coefficients p_j depend on the external loads parameters and on the elements of damping matrix $[d_{jk}]$. When damping is absent ($\varepsilon \equiv 0$) and at $\beta < \beta_*$ the equation

$$g_n(s^2) = p_0 s^{2n}+p_2 s^{2n-2}+p_4 s^{2n-4}+\ldots+p_{2n} = 0, \tag{6.11}$$

has purely imaginary and simple roots, and $p_{2n} \neq 0$. At $\beta < \tilde{\beta}_*$ each root of the equation (6.1) is an analytic function of the parameter ε. An asymptotic expansion of this function is

$$s(\varepsilon) = s_0 + a_1\varepsilon + o(\varepsilon),$$

where $s_0 = s(0)$, $a_1 = \mathrm{d}s(0)/\mathrm{d}\varepsilon$. According to the theorem on implicit functions we obtain from (6.10)

$$\frac{\mathrm{d}s(0)}{\mathrm{d}\varepsilon} = -\frac{s_0(p_1 s_0^{2n-2} + p_3 s_0^{2n-4} + \dots + p_{2n-3}s_0^2 + p_{2n-1})}{\left.\dfrac{\partial P_{2n}}{\partial s}\right|_{\varepsilon=0}}.$$

Putting the coefficient a_1 to zero we get

$$f_{n-1}(s_0^2) = p_1 s_0^{2n-2} + p_3 s_0^{2n-4} + \cdots + p_{2n-3}s_0^2 + p_{2n-1} = 0.$$

As the quantities s_0^2 are the roots of the equation (6.11), so calculations reduce to construction of the resultant of two polynomials f_{n-1} and g_n. From the polynomials theory it is known that

$$R(f_{n-1}, g_n) = \begin{vmatrix} p_1 & p_3 & p_5 & \cdots & p_{2n-1} & 0 & \cdots & 0 \\ 0 & p_1 & p_3 & \cdots & p_{2n-3} & 0 & \cdots & 0 \\ 0 & 0 & p_1 & \cdots & p_{2n-5} & 0 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \\ p_0 & p_2 & p_4 & \cdots & p_{2n-2} & p_{2n} & \cdots & 0 \\ 0 & p_0 & p_2 & \cdots & p_{2n-4} & p_{2n-2} & \cdots & 0 \\ 0 & 0 & p_0 & \cdots & p_{2n-6} & p_{2n-4} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} \begin{matrix} \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n \\ = \\ \left.\vphantom{\begin{matrix}a\\a\\a\end{matrix}}\right\} n-1 \end{matrix}$$

$$= \begin{vmatrix} p_1 & p_3 & p_5 & \cdots & p_{2n-1} & 0 & \cdots & 0 \\ p_0 & p_2 & p_4 & \cdots & p_{2n-2} & p_{2n} & \cdots & 0 \\ 0 & p_1 & p_3 & \cdots & p_{2n-3} & p_{2n-1} & \cdots & 0 \\ 0 & p_0 & p_2 & \cdots & p_{2n-4} & p_{2n-2} & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots \end{vmatrix} = \Delta_{2n-1}$$

where Δ_{2n-1} is a Hurwitz' determinant of the $2n-1$ order corresponding to a polynomial obtained from the polynomial (6.10) by an obvious procedure. Thus the equation of the critical flutter surface when damping is infinitesimal, has the form $\Delta_{2n-1} = 0$.

7. EXAMPLE OF A CONTINUOUS DAMPED SYSTEM

Let us consider a problem on stability of a cantilever bar subjected at the free end to a tangential force P and a dead load Q (Fig. 3). The bar is made of the linear standard viscoelastic material with the deformation law as follows

$$L_1(\sigma) = L_2(e)$$

Here

$$L_1 = \frac{\tau}{E_0}\frac{\partial}{\partial t} + \frac{1}{E_\infty}, \qquad L_2 = \tau\frac{\partial}{\partial t} + 1$$

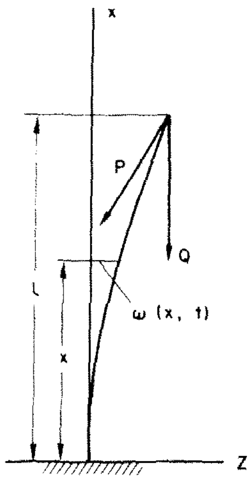

FIG. 3. Cantilever subjected to follower and dead forces.

σ—stress, e—strain, τ—relaxation time, E_0 and E_∞—unrelaxed and relaxed moduli, respectively. For the investigation of stability the dynamic method [3] is used. Equation of small vibration of the viscoelastic bar near the equilibrium position is obtained using the elastic–viscoelastic analogy. The elasticity modulus in equation written for elastic bar is to substitute by a complex modulus, $E^* = L_2(s)/L_1(s)$ where s is the characteristic exponent. As a result we get the following nonself-adjoint boundary value problem

$$(1+\eta s)\frac{d^4W}{d\xi^4}+(\alpha+\beta)(1+\gamma\eta s)\frac{d^2W}{d\xi^2}+(s^2+\gamma\eta s^3)W = 0 \tag{7.1}$$

$$W = \frac{dW}{d\xi} = 0 \quad \text{at} \quad \xi = 0$$

$$\frac{d^2W}{d\xi^2} = 0 \quad \text{at} \quad \xi = 1 \tag{7.2}$$

$$\frac{d^3W}{d\xi^3}+\alpha\frac{1+\gamma\eta s}{1+\eta s}\frac{dW}{d\xi} = 0 \quad \text{at} \quad \xi = 1.$$

Here $W(\xi)$—vibration mode,

$$\xi = \frac{x}{l}, \qquad \beta = \frac{Pl^2}{E_\infty J}, \qquad \alpha = \frac{Ql^2}{E_\infty J}, \qquad \eta = \Omega_0\tau, \qquad \Omega_0^2 = \frac{E_\infty J}{ml^4}.$$

J—inertia moment of the cross section, l—length, m—mass per length unit, $\gamma = E_\infty/E_0$, $0 \le \gamma < 1$. The parameter η characterizes the energy dissipation. Putting $\gamma = 0$ we obtain the Voigt material. The parameters α and β are positive if the corresponding forces produce compression of the bar. Further a case $\alpha > 0$ and $\beta > 0$ is considered.

A solution of the equation (7.1) has a form

$$W(\xi) = \sum_{j=1}^{4} C_j e^{\lambda_j\xi},$$

where λ_j ($j = 1, 2, 3, 4$) are roots of the equation

$$(1+\eta s)\lambda^4+(\alpha+\beta)(1+\gamma\eta s)\lambda^2+(s^2+\gamma\eta s^3) = 0. \tag{7.3}$$

Satisfying the boundary conditions (7.2) a set of linear algebraic equations for C_j is obtained. Condition of existence of nontrivial solution yields

$$\begin{vmatrix} 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 \\ \lambda_1^2\,e^{\lambda_1} & \lambda_2^2\,e^{\lambda_2} & \lambda_3^2\,e^{\lambda_3} & \lambda_4^2\,e^{\lambda_4} \\ (\lambda_1^3+\psi\lambda_1)\,e^{\lambda_1} & (\lambda_2^3+\psi\lambda_2)\,e^{\lambda_2} & (\lambda_3^3+\psi\lambda_3)\,e^{\lambda_3} & (\lambda_4^3+\psi\lambda_4)\,e^{\lambda_4} \end{vmatrix} = 0 \tag{7.4}$$

$$\psi = \alpha\frac{1+\gamma\eta s}{1+\eta s}.$$

The roots of the equation (7.3) are

$$\lambda_1 = (1+\eta s)^{-\frac{1}{4}}\left\{-\frac{\alpha+\beta}{2}(1+\gamma\eta s)+\left[\frac{(\alpha+\beta)^2}{4}(1+\gamma\eta s)^2-(1+\eta s)(s^2+\gamma\eta s^3)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\lambda_2 = i(1+\eta s)^{-\frac{1}{4}}\left\{\frac{\alpha+\beta}{2}(1+\gamma\eta s)+\left[\frac{(\alpha+\beta)^2}{4}(1+\gamma\eta s)^2-(1+\eta s)(s^2+\gamma\eta s^3)\right]^{\frac{1}{2}}\right\}^{\frac{1}{2}}$$

$$\lambda_3 = -\lambda_1, \qquad \lambda_4 = -\lambda_2.$$

Expanding the determinant (7.4) we obtain the following characteristic equation:

$$\begin{aligned} F(\lambda_1, \lambda_2, \psi) = {} & \lambda_1^4+\lambda_2^4+\lambda_1\lambda_2(\lambda_1^2+\lambda_2^2)\,\mathrm{sh}\lambda_1\,\mathrm{sh}\lambda_2-2\lambda_1^2\lambda_2^2\,\mathrm{ch}\lambda_1\,\mathrm{ch}\lambda_2 \\ & +\psi[\lambda_1^2+\lambda_2^2-(\lambda_1^2+\lambda_2^2)\,\mathrm{ch}\lambda_1\,\mathrm{ch}\lambda_2+2\lambda_1\lambda_2\,\mathrm{sh}\lambda_1\,\mathrm{sh}\lambda_2] = 0. \end{aligned} \tag{7.5}$$

Another form of this equation is

$$\begin{aligned} F(s, \eta, \alpha, \beta) = {} & \frac{1+\gamma\eta s}{1+\eta s}(\alpha+\beta)^2-2s^2(1+\mathrm{ch}\lambda_1\,\mathrm{ch}\lambda_2) \\ & +s(\alpha-\beta)\left(\frac{1+\gamma\eta s}{1+\eta s}\right)^{\frac{1}{2}}\mathrm{sh}\lambda_1\,\mathrm{sh}\lambda_2 \\ & -\alpha(\alpha+\beta)\frac{1+\gamma\eta s}{1+\eta s}(1-\mathrm{ch}\lambda_1\,\mathrm{ch}\lambda_2) = 0. \end{aligned} \tag{7.6}$$

Let us consider divergence instability. Setting in the equation (7.6) $s = 0$ yields

$$\alpha\cos\sqrt{(\alpha+\beta)}+\beta = 0. \tag{7.7}$$

On the α, β-plane this equation determines boundaries of the divergence quasistability region. As was mentioned above, the singular points divide this line to stability and instability segments. Singular points are to satisfy the equation

$$a_{11} = \frac{\partial^2 F(0, 0, \tilde{\alpha}_*, \tilde{\beta}_*)}{\partial s\,\partial\eta} = 0.$$

From the equation (7.6) we obtain

$$\beta_* + \alpha_* \cos\sqrt{(\alpha_*+\beta_*)} - \alpha_*\sqrt{(\alpha_*+\beta_*)}\sin\sqrt{(\alpha_*+\beta_*)} = 0. \tag{7.8}$$

As $\tilde{\alpha}_*$ and $\tilde{\beta}_*$ satisfy the equation (7.7), we get from equation (7.8):

$$\alpha_* + \beta_* = \pi^2. \tag{7.9}$$

This straight line on the α, β-plane intersects the curve (7.7) in a single point ($\alpha_* = \beta_* = \pi^2/2$). Thus a point ($\alpha_* = \beta_* = \pi^2/2$) is a single singular point on the boundary of divergence instability. Using considerations from Sect. 2 and 3 we conclude, that the segment of this line adjoining to the α-axis is a stability boundary. The rest of this line belongs to the instability region.

Let us consider the flutter instability. From equation (7.5) we find that

$$a_{01} = \sum_{j=1,2} \frac{\partial F}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial \eta} + \frac{\partial F}{\partial \psi}\alpha s_*(\gamma - 1), \tag{7.10}$$

and

$$a_{10} = \sum_{j=1,2} \frac{\partial F}{\partial \lambda_j}\frac{\partial \lambda_j}{\partial s} = 0, \tag{7.11}$$

according to condition. Here a zero above the letters denotes that these expressions are taken at $\eta = 0$, $s = s_*$, $\alpha = \tilde{\alpha}_*$, $\beta = \tilde{\beta}_*$. As all the roots of the equation (7.3) are different

$$\frac{\partial \lambda_j}{\partial s} = -\frac{2s_*}{4\lambda_j^3 + 2(\tilde{\alpha}_* + \tilde{\beta}_*)\lambda_j}, \qquad \frac{\partial \lambda_j}{\partial \eta} = -\frac{(1-\gamma)\lambda_j^4 s_*}{4\lambda_j^3 + 2(\tilde{\alpha}_* + \tilde{\beta}_*)\lambda_j}, \tag{7.12}$$

$$\frac{\partial \lambda_j}{\partial \eta} = \frac{\partial \lambda_j}{\partial s}\frac{(1-\gamma)\lambda_j^4}{2} (j = 1, 2).$$

Taking into account relations (7.12) we may factor out of the brackets the complex factor and consider the sums as a scalar product of appropriate vectors in three-dimensional space. This consideration yields that the quantity $a_{01} \neq 0$ and that it is purely imaginary at arbitrary $0 \leq \gamma < 1$ and $s_* \neq 0$. Hence $\alpha_* < \tilde{\alpha}_*$ or $\beta_* < \tilde{\beta}_*$ at $\eta \to 0$ and at arbitrary γ.

The equation of the boundary of flutter instability at $\eta \to 0$ is to be obtained by the elimination of ω from the following set of equations [see equation (5.2) also]:

$$\begin{aligned} &\beta(\alpha+\beta) + [r_1^3(\alpha^2+\alpha\beta+2\omega^2) - \omega(\alpha-\beta)r_2^3]\left[\left(\frac{\alpha+\beta}{2}\right)^2 + \omega^2\right]^{-\frac{1}{2}} \\ &\quad \times \operatorname{ch} r_1 \cos r_2 - \left[\left(\frac{\alpha+\beta}{2}\right)^2 + \omega^2\right]^{-\frac{1}{2}} [r_2^3(\alpha^2+\alpha\beta+2\omega^2) \\ &\quad + \omega(\alpha-\beta)r_1^3] \operatorname{ch} r_1 \sin r_2 - \tfrac{1}{2}\omega(\alpha-\beta) \operatorname{sh} r_1 \sin r_2 \\ &\quad + \alpha(\alpha+\beta) \operatorname{ch} r_1 \cos r_2 = 0 \end{aligned} \tag{7.13}$$

$$\beta(\alpha+\beta) + 2\omega^2 - \omega(\alpha-\beta)\operatorname{sh} r_1 \sin r_2 + [\alpha(\alpha+\beta) + 2\omega^2]\operatorname{ch} r_1 \cos r_2 = 0$$

$$r_1^2 = -\frac{\alpha+\beta}{2} + \left[\left(\frac{\alpha+\beta}{2}\right)^2 + \omega^2\right]^{\frac{1}{2}}$$

$$r_2^2 = \frac{\alpha+\beta}{2} + \left[\left(\frac{\alpha+\beta}{2}\right)^2 + \omega^2\right]^{\frac{1}{2}}.$$

8. METHOD OF NUMERICAL ANALYSIS

The behaviour of the characteristic exponents when one of the parameters varies has been investigated numerically. Solution of the transcendental equations has been reduced to a Cauchy's problem for a set of two ordinary differential equations of the first order. The general idea is as follows. Setting in equation (5.5) $s = \xi + i\psi$ and dividing real and imaginary parts yields:

$$F_I(\xi, \psi, \eta, \alpha, \beta) = 0 \qquad F_R(\xi, \psi, \eta, \alpha, \beta) = 0. \tag{8.1}$$

After the differentiation of the equations (8.1) with respect to β we obtain

$$\frac{\partial F_I}{\partial \xi}\frac{d\xi}{d\beta} + \frac{\partial F_I}{\partial \psi}\frac{d\psi}{d\beta} = -\frac{\partial F_I}{\partial \beta} \qquad \frac{\partial F_R}{\partial \xi}\frac{d\xi}{d\beta} + \frac{\partial F_R}{\partial \psi}\frac{d\psi}{d\beta} = -\frac{\partial F_R}{\partial \beta}. \tag{8.2}$$

Transforming this set of equations to the normal form (this procedure is always admissible as $\eta \neq 0$) we get the following Cauchy problem:

(8.3)

$$\frac{d\xi}{d\beta} = \frac{\dfrac{\partial F_I}{\partial \beta}\dfrac{\partial F_R}{\partial \psi} - \dfrac{\partial F_I}{\partial \psi}\dfrac{\partial F_R}{\partial \beta}}{J(F_I, F_R)} \qquad \frac{d\psi}{d\beta} = \frac{\dfrac{\partial F_I}{\partial \xi}\dfrac{\partial F_R}{\partial \psi} - \dfrac{\partial F_I}{\partial \beta}\dfrac{\partial F_R}{\partial \xi}}{J(F_I, F_R)} \tag{8.3}$$

$$\xi_j(\beta_0) = \xi_{j0} \qquad \psi_j(\beta_0) = \psi_{j0} \tag{8.4}$$

where

$$J(F_I, F_R) = \frac{\partial F_I}{\partial \psi}\frac{\partial F_R}{\partial \xi} - \frac{\partial F_I}{\partial \xi}\frac{\partial F_R}{\partial \psi}$$

The Cauchy problem has been solved numerically by a digital computer using Runge-Kutta's procedure. Partial derivatives in the equation (8.3) were substituted by finite difference expressions. Initial dates were computed by the gradient method. Instead of equation (7.6) the following equation was considered:

$$\Phi(\xi, \psi, \eta, \alpha, \beta) = |F(s, \eta, \alpha, \beta)|^2 = 0. \tag{8.5}$$

Successive approximations for roots of the equation (8.5) were calculated using a formula

$$z_{k+1} = z_k - \lambda_k \operatorname{grad} \Phi(z_k, \eta, \alpha, \beta) \tag{8.6}$$

where vector $z_k = \{\xi_k, \psi_k\}$ and

$$\lambda_k = \frac{\Phi(z_k, \eta, \alpha, \beta)}{|\operatorname{grad} \Phi(z_k, \eta, \alpha, \beta)|^2}.$$

Computation of the boundaries of the instability regions on the α, β-plane at different values of η and γ were performed by the same method. The critical value β_* of the parameter at fixed values of the parameters α, γ and η was determined as a minimal root of the equation (8.5) at $\xi = 0$. Evaluation of the boundaries of quasistability regions and stability regions at $\eta \to 0$ were made by similar procedure. Calculations were realized by computer BESM-2M.

9. DISCUSSION OF RESULTS

The instability regions on α, β-plane for different values of damping parameters γ and η are presented on Figs. 4–7. Here by broken lines the boundaries of quasistability regions corresponding to the case $\eta \equiv 0$ are plotted.

Let us discuss in detail Fig. 4 corresponding to the case $\gamma = 0$ (the bar is made of Voigt's material). Both the stability and quasistability regions are limited by lines of two types: the lines intersection of which is accompanied by vibrational instability, and the lines associated with unvibrational instability. The Fig. 4 shows that the introduction of infinitesimal damping ($\eta \to 0$) transforms a considerable part of the quasistability region in the instability region. For example, when $\alpha = 0$ (the bar is subjected to the follower

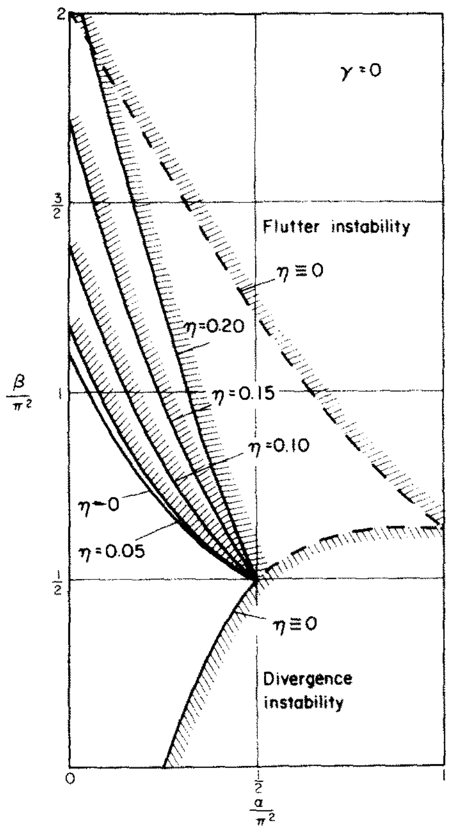

FIG. 4. Instability and quasistability regions at $\gamma = 0$ and at different values of η.

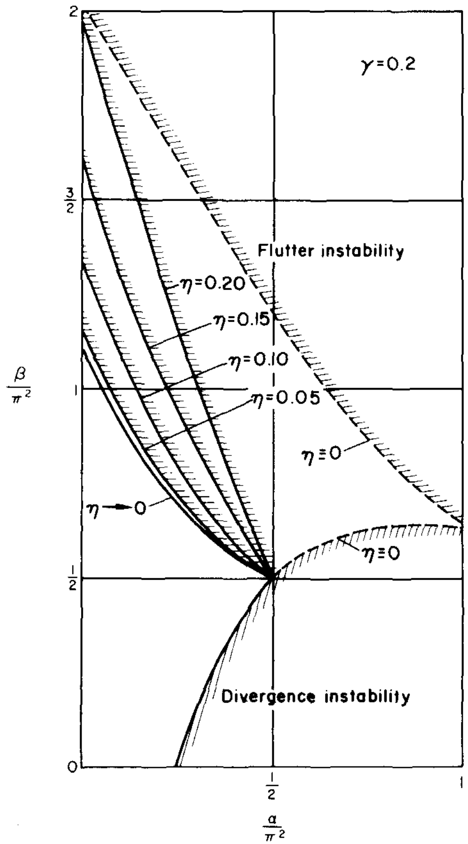

FIG. 5. The same at $\gamma = 0{\cdot}2$.

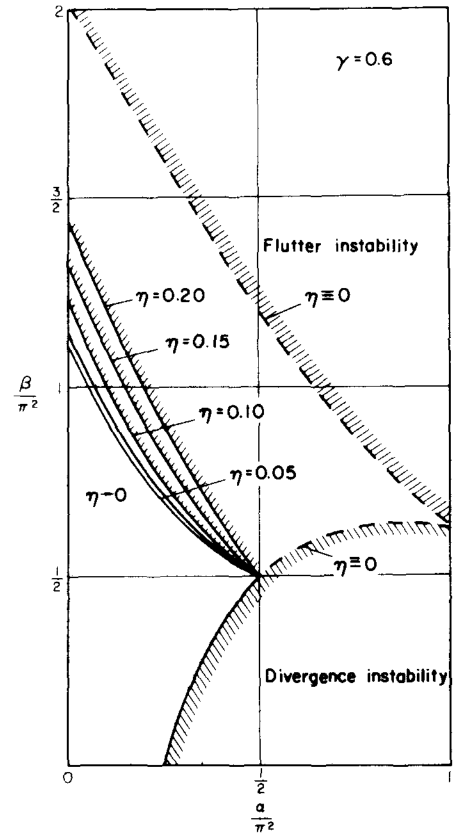

FIG. 6. The same at $\gamma = 0{\cdot}6$.

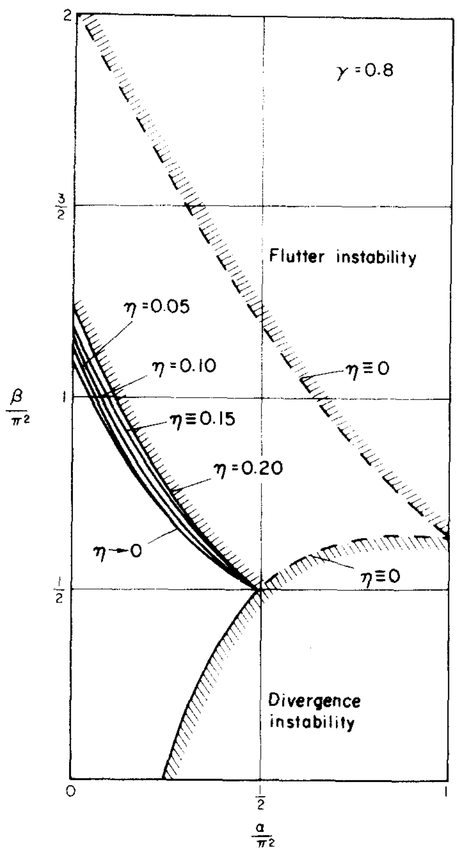

FIG. 7. The same at $\gamma = 0{\cdot}8$.

force only) the critical parameter $\beta_* = 10{\cdot}94$ is approximately twice less than quasi-critical value $\tilde{\beta}_* = 20{\cdot}05$. This fact has been mentioned earlier too [3, 6, 11]. When damping parameter η increases, the stability region is gradually widening. But even at $\eta = 0{\cdot}20$ a considerable part of the quasistability regions belongs in fact to the instability region.

There is an interesting fact in the presence of a singular point at $\alpha = \beta = \pi^2/2$. At this point the vibrational instability curves corresponding to various values of η and the unvibrational instability curve intersect. Another interesting fact is that the stability and quasistability regions are unconvex (it is known [20] that stability regions for elastic systems subjected to conservative forces are convex). Unconvexity of stability region of the panel flutter problem has been mentioned by Bolotin [3].

In connection with the question about the unconvexity it is appropriate to remember a phenomenon of the jump of the critical force discovered by Dzhanelidze [17]. Dzhanelidze has considered a problem of the stability of a cantilever with a concentrated mass on the end compressed by a dead and follower forces. Damping was not taken into account. Plotting the sum $P+Q$ corresponding to the boundary of the quasistability region against to ratio P/Q, Dzhanelidze has found a jump at $P/Q = 1$. This jump corresponds to the transition from the static instability line to the vibrational instability line. The jump phenomenon in the double pendulum problem was discussed by Herrmann and

Bungay [21]. An analogous result was obtained in a case of the distributed mass (see Fig. 8). Some authors state that this fact is a consequence of the oversimplification of the problem, and they suppose that the jump will be eliminated when damping is introduced. In fact in presence of damping the jump of the $\alpha+\beta$, β/α plane vanishes (Fig. 8). But the real cause of the jump is unconvexity of the stability region. The unconvexity remains in the presence of damping (Figs. 4–7). Therefore, replacing the parameters α, β by its independent combination we can find out the jump phenomenon too. The jump on $\alpha+\beta$, β/α plane disappears because of the coincidence of two points: the contact point of the line $\alpha = \beta$ with the static instability curve and the singular point.

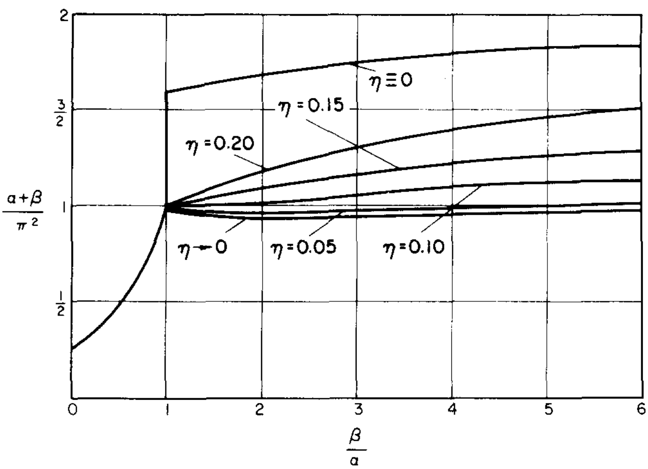

FIG. 8. Summary critical force as a function of the ratio of follower and dead forces.

The diagrams presented on the Figs. 4–7 show a gradual alteration of the stability regions when the parameter γ increases from zero to $\gamma = 0{\cdot}8$. The topology of the stability regions does not vary. But the increasing of γ causes a weaker dependence of the curve of the vibrational instability on the quantity η. Considering partial damping coefficients corresponding to the two first natural modes we obtain a qualitative explanation of this phenomenon. Rigorously, a concept of partial damping coefficients for a dissipatively bounded system is conventional. We shall express these coefficients through the power of damping forces when displacements coincide with natural modes of an elastic bar and the frequencies of motion coincide with natural frequencies. Thus

$$d_{kk} = \frac{E_I(\omega_k)}{E_R(\omega_k)}$$

where $E_R(\omega_k)$ and $E_I(\omega_k)$ are real and imaginary parts of the complex modulus at the natural frequency ω_k.

The ratio of two first partial damping coefficients d_{22}/d_{11} is plotted on Fig. 9 against the parameters γ and η. The diagram shows that when the parameter γ increases, the ratio d_{22}/d_{11} increases too. Using an analogy with finite-degree-of-freedom systems (see for

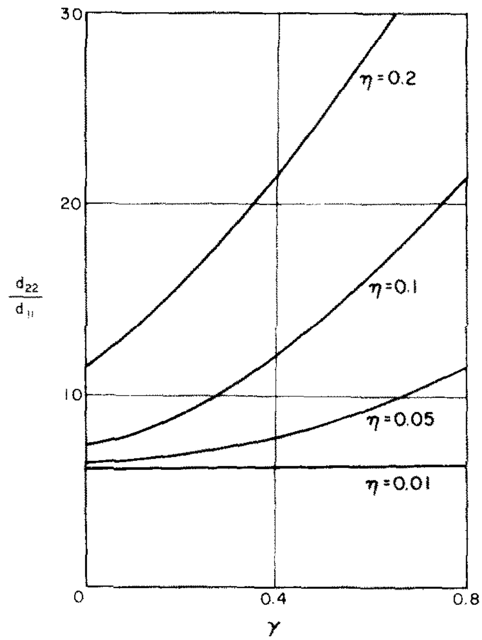

FIG. 9. Partial damping coefficient as functions of the value γ.

example Fig. 2) we can suppose that the increasing of γ causes the decreasing of the critical parameters. Results of direct computations are in agreement with this statement.

A next question to be discussed is the following one. Intuitive considerations [7] let us suppose that in spite of rigorous theory the quasicritical parameters have some physical and engineering sense. When damping is sufficiently small, the exceeding of the quasicritical value results in some variation of the behaviour of the physical system. To investigate this phenomenon, the properties of the characteristic exponents will be studied in the case of sufficiently small damping.

The real and imaginary parts of the two first characteristic exponents s_1, s_2 as functions of the follower force parameter β at $\alpha = 0$, $\gamma = 0$ and at different values of η are presented in Fig. 10. For comparison on the same diagram the characteristic exponents calculated at $\eta \equiv 0$ are presented. When η is very small (for example when $\eta = 0{\cdot}001$) the characteristic exponents differ a little from the calculated for the case $\eta \equiv 0$. But alteration of the sign of the real part $\operatorname{Re} s_1$ occurs at $\beta < \tilde{\beta}_*$. It is essentially, that when η is very small the increment $\operatorname{Re} s_1$ is sufficiently small in the range $\beta_* < \beta < \tilde{\beta}_*$. The rapid growing of the increment begins only when $\beta > \tilde{\beta}_*$. Hence, although at very small η the exceeding of the critical value β_* leads to the instability, but the sharp growing begins only, when exceeding of the quasicritical value $\tilde{\beta}_*$ occurs. On the Fig. 11 the increment $\operatorname{Re} s_1$ is plotted against β on a large scale ($\eta = 0{\cdot}001$). An analogous diagram for a two-degrees-of-freedom system was presented in the paper by Herrmann and Jong [9].

10. CONCLUDING REMARKS

In this paper a question about relations between stability and quasistability regions was considered both from a general point of view and on numerical examples. A method of expansion in fractional powers of parameters is applied. A principal conclusion is: for real laws of damping a considerable part of quasistability region belongs to the instability region. From this rigorous point of view the majority of papers dealing with stability of nonconservative systems (including panel flutter) need to be reconsidered.

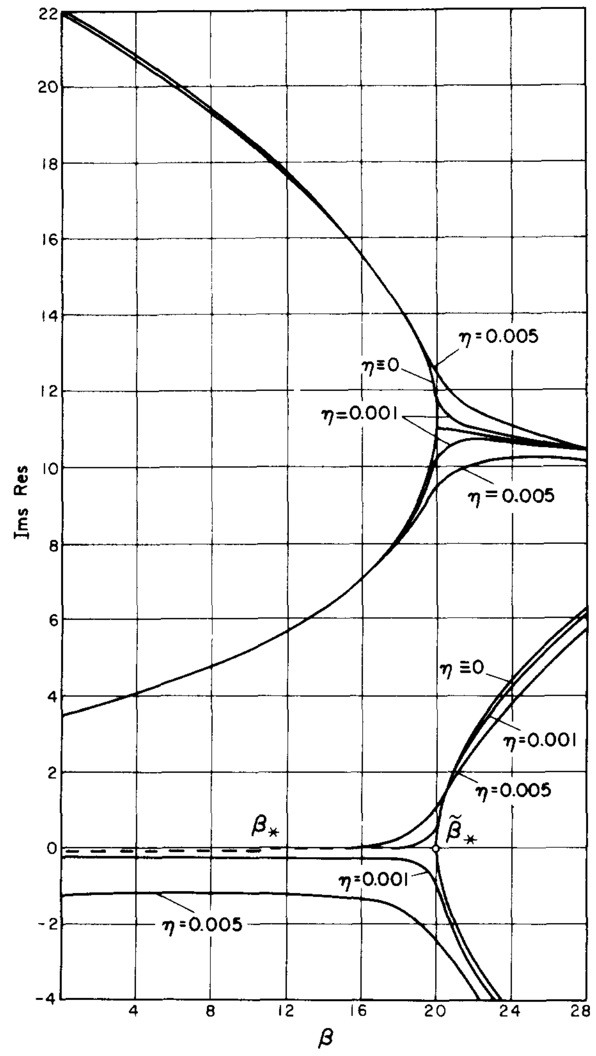

FIG. 10. Real and imaginary parts of the characteristic exponents s_1 as functions of the parameter β at different η.

Only in a case of very small damping may we expect that for mutual parts of the stability and quasistability regions a "quiet" flutter is typical and for the vibrational instability region (in proper sense)—a "violent" flutter. It is a hypothesis that has to be confirmed by solution of nonlinear problems and by experiments.

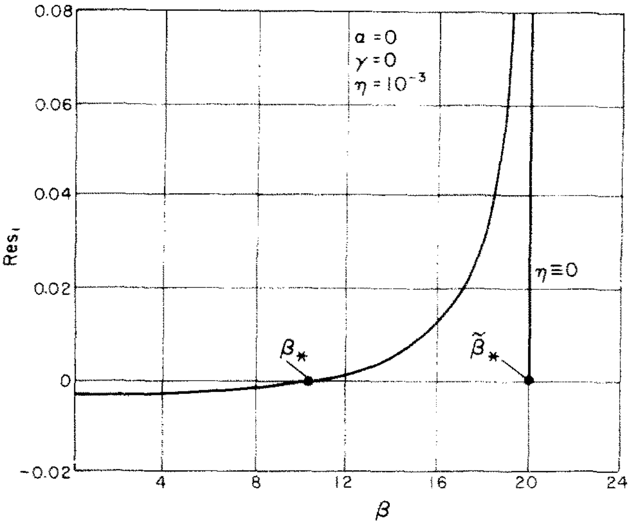

FIG. 11. Real part of the characteristic exponent s_1 as a function of the parameter β at $\eta = 0{\cdot}001$.

REFERENCES

[1] Е. Л. Николаи, Об устойчивости прямолинейной формы равновесия сжатого и скрученного стержня, Изв. Ленингр. политехн. ин-та, **31** (1928). См. также: Е. Л. Николаи, Труды по механике, Гостехиздат, (1955).

[2] Е. Л. Николаи, К вопросу об устойчивости скрученного стержня, Вестн. прикл. матем. мех., **1** (1929).

[3] В. В. Болотин, Неконсервативные задачи теории упругой устойчивости, Физматгиз, (1961) [английский перевод—Пергамон Пресс (1963)].

[4] G. HERRMANN, Stability of equilibrium of elastic systems subjected to nonconservative forces. *Appl. Mech. Rev.* **20** (1967).

[5] Г.Ю. Джанелидзе, Устойчивость упругих систем при динамических нагрузках, Сб. "Проблемы устойчивости в строительной механике", Стройиздат (1963).

[6] H. ZIEGLER, Die Stabilitätskriterien der Elastomechanik. *Ing.-Arch.* **20**, (1) 49 (1952).

[7] В. В. Болотин, О колебаниях и устойчивости стержней, находящихся под действием неконсервативных сил, Сб. "Колебания в турбомашинах", Изд-во АН СССР (1959).

[8] М. Я. Леонов и Л. М. Зорий, О влиянии трения на критическую силу сжатого стержня, Доклады АН СССР, т. **145**, (2) 295, (1962).

[9] G. HERRMANN and I.-C. JONG. On the destabilizing effect of damping in nonconservative elastic systems. *J. appl. Mech.* **32**, 592 (1965).

[10] H. LEIPHOLZ, Über den Einfluß der Dämpfung bei nichtkonservativen Stabilitätsproblemen elastischer Stäbe. *Ing.-Arch.* **33**, 308 (1964).

[11] S. NEMAT-NASSER, S. N. PRASAD and G. HERRMANN, Destabilizing effect of velocity-dependent forces in nonconservative continuous systems. *AIAA Jl.* **4**, 1276 (1966).

[12] S. NEMAT-NASSER and G. HERRMANN, Some general considerations concerning the destabilizing effect in nonconservative systems. *Z. angew. mech. phys.* **17**, 305 (1966).

[13] Н. И. Жинжер, О дестабилизирующем влиянии трения на устойчивость неконсервативных упругих систем. Механика твердого тела, (3) (1968).

[14] В. В. Болотин, О понятии устойчивости в строительной механике. Сб. "Проблемы устойчивости в строительной механике", Стройиздат (1963).

[15] А. А. Мовчан, О прямом методе Ляпунова в задачах устойчивости упругих систем. ПММ, **23**, (3) (1959).

[16] W. T. KOITER, The energy criterion of stability for continuous elastic bodies, *Proc. R. Acad. Sci. Holland,* **B68** (1965).

[17] Г.Ю. Джанелидзе, Об устойчивости стержня под действием следящей силы. Труды Ленинградского политехнического института, (192) (1958).

[18] А. А. Маркушевич, Теория аналитических функций, т. 1, Изд-во "Наука" (1967).

[19] Н. Г. Чеботарев, Собрание сочинений, т. 3, Изд-во АН СССР (1950).
[20] П. Ф. Папкович, Несколько общих теорем, относящихся к устойчивости упругих систем. Труды Ленинградского политехнического института, вып. 1 (1937). См. также П. Ф. Папкович, Строительная механика корабля, ч. II, Судпромгиз, (1941).
[21] G. Herrmann and R. W. Bungay, On the stability of elastic systems subjected to nonconservative forces. *J. appl. Mech.* **31**, (3) (1964).
[22] G. Herrmann and I. C. Jong, On nonconservative stability problems of elastic systems with slight damping. *J. appl. Mech.* **33**, (1) (1966).
[23] T. B. Benjamin, The threefold classification of unstable disturbances in flexible surfaces bounding inviscid flow. *J. Fluid Mech.* **16** (1963).
[24] S. Nemat-Nasser, On the stability of the equilibrium of nonconservative continuous systems with slight damping. *J. appl. Mech.* **34** (1967).

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Резюме—Рассматривается вопрос о соотношении областей устойчивости и квазиустойчивости для упругих и вязкоупругих систем, нагруженных неконсервативными силами. На основе метода разложения по дробным степеням параметров строго доказаны утверждения, которые ранее высказывались в ряде работ на основании полуинтуитивных соображений, аргументации по аналогии и неполной индукции. В первой части статьи при общих предположениях доказывается ряд утверждений как для распределенных так и для дискретных систем. Показывается, что для реальных законов демпфирования значительная частъ области квазиустойчивости в действительности принадлежит области неустойчивости. С этой точки зрения подавляющее большинство работ по устойчивости неконсервативных упругих систем (включая работы по панельному флаттеру) нуждается в пересмотре. Для иллюстрации общих утверждений во второй части статъи приводится численное исследование устойчивости консольного стержня из линейного стандартного вязко-упругого материала, нагруженного следящей и мертвой силами. Обсуждается ряд явлений, присущих неконсервативным вязко-упругим системам.